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THE STABILIZATION OF THE EQUILIBRIUM OF CONSERVATIVE SYSTEMS USING GYROSCOPIC FORCES†

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Criteria for the instability of the equilibrium of gyroscopically coupled systems, when the gyroscopic forces may be predominant, are presented. It is shown that the clear predominance of gyroscopic forces over potential forces still does not ensure stability of the equilibrium. The structure of the potential forces remains the key here. As an example, the problem of the stability of the steady-state motions of an artificial satellite is considered. © 2000 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Consider the holonomic system, with n degrees of freedom

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}}-\frac{\partial L}{\partial \mathbf{q}}=\mathbf{0} \tag{1.1}$$

the Lagrangian of which has the form

$$L(\mathbf{q},\dot{\mathbf{q}})=L_2(\mathbf{q},\dot{\mathbf{q}})+L_1(\mathbf{q},\dot{\mathbf{q}})+L_0(\mathbf{q})=\tfrac{1}{2}\dot{\mathbf{q}}^TA(\mathbf{q})\dot{\mathbf{q}}+\mathbf{f}(\mathbf{q})^T\dot{\mathbf{q}}+L_0(\mathbf{q}) \tag{1.2}$$

We shall assume that $L(\mathbf{q}, \dot{\mathbf{q}}) \in C^2(D_\mathbf{q}\times R^n)$, the quadratic form $L_2(\mathbf{0}, \dot{\mathbf{q}})$ is positive definite and the point $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ corresponds to the equilibrium position of system (1.1), (1.2), and $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, $L_0(\mathbf{0}) = 0$.

It is well known [1, 2] that the problem of the stability of steady motions reduces to Eqs (1.1) with Lagrangian (1.2). The criteria for the instability of such motions, that is, of the equilibria of system (1.1), (1.2), is usually referred to as the inverse of Routh's theorem [1, 3].

If the function $\mathbf{q} = \mathbf{0}$ has a strict local maximum at the point $L_0(\mathbf{q})$, then, by Routh's theorem [4], the steady motion (the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$) of system (1.1), (1.2)) is stable. If, however, there is no maximum, then stability caused, in particular, by the stabilizing action of gyroscopic forces, as well as instability can occur [5, 6]. In this case, the issue of the gyroscopic stabilization of the equilibrium of system (1.1), (1.2) remains somewhat unclear.

2. THE INSTABILITY OF EQUILIBRIUM UNDER CONDITIONS WHEN GYROSCOPIC FORCES ACT

We shall assume that generalized coordinates are chosen in such a way that

$$L_0(\mathbf{q})=L_0^{(2)}(\mathbf{q})+L_0^{(m)}(\mathbf{q})+R(\mathbf{q}),\quad R(\mathbf{q})=o(\|\mathbf{q}\|^m) \tag{2.1}$$

$$L_0^{(2)}(\mathbf{q})=\frac{1}{2}\sum_{k=3}^{n}\lambda_k q_k^2,\quad \text{const}=\lambda_k<0 \tag{2.2}$$

where $L_0^{(m)}(\mathbf{q})$ is a homogeneous form of degree $m > 2$.

We will denote by $\hat{\Psi}(\mathbf{q})$ the restriction of an arbitrary function $\Psi(\mathbf{q})$, including a vector function, to the set of zeros of the quadratic form $L_0^{(2)}(\mathbf{q})$.

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Theorem 1. Suppose that, together with the initial assumptions regarding the Lagrangian L, equalities (2.1) and (2.2) hold and, furthermore,

1) $\partial \widehat{L_0^{(m)}}/\partial \hat{\mathbf{q}} \neq \mathbf{0}, \quad \forall \hat{\mathbf{q}} \in R^2 \setminus \{0\}$;

2) the function $\widehat{L_0^{(m)}}(\mathbf{q})$ does not have an extremum at the point $q_1 = q_2 = 0$;
3) $\det G^2(\mathbf{0}) \neq 0$, where $G^2(\mathbf{q})$ is a 2×2 matrix, obtained from the matrix

$$G(\mathbf{q}) = (g_{ij}(\mathbf{q})), \quad g_{ij}(\mathbf{q}) = (\partial f_i/\partial q_j - \partial f_j/\partial q_i), \quad i, j = 1, \ldots, n$$

by cancelling the last $n-2$ rows and the last $n-2$ columns;

4) $\lim\limits_{\|\mathbf{q}\| \to 0} \|\partial R/\partial \mathbf{q}\| (\|\mathbf{q}\|)^{-m+1-\alpha} = 0, \quad \text{const} = \alpha > 0.$

The equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of system (1.1), (1.2) is then unstable.

Proof. Making the substitution

$$\partial L_2/\partial \dot{\mathbf{q}} = A\dot{\mathbf{q}} = \mathbf{p}$$

in (1.1), we arrive at the system of equations

$$\dot{\mathbf{q}} = \partial H/\partial \mathbf{p}, \quad \dot{\mathbf{p}} = -\partial H/\partial \mathbf{q} - GA^{-1}\mathbf{p} \tag{2.3}$$

$$H(\mathbf{q}, \mathbf{p}) = \tfrac{1}{2}\mathbf{p}^T A^{-1}(\mathbf{q})\mathbf{p} - L_0(\mathbf{q}) = h = \text{const} \tag{2.4}$$

By Condition 2 of Theorem 1, the set

$$\Omega\{(\mathbf{q}, \mathbf{p}) \in s_\varepsilon = \{(\mathbf{q}, \mathbf{p}) \in D_{\mathbf{q}} \times R^n, \ \|\mathbf{q} \oplus \mathbf{p}\| < \varepsilon\} : H(\mathbf{q}, \mathbf{p}) = h \leqslant 0\}$$

is a non-empty set.

From equality (2.4), taking account of relations (2.1) and (2.2), we have

$$\|\mathbf{p}\|^2 < \lambda \|\mathbf{q}\|^m, \quad \forall (\mathbf{q}, \mathbf{p}) \in \Omega, \quad 0 < \lambda = \text{const} \tag{2.5}$$

Noting that the inequality

$$-L_0^{(2)}(\mathbf{q}) \leqslant L_0^{(m)}(\mathbf{q}) + R(\mathbf{q}) \tag{2.6}$$

holds for the set Ω, as a consequence we obtain

$$\sum_{k=3}^{n} q_k^2 \leqslant \mu (q_1^2 + q_2^2)^{m/2}, \quad \text{const} = \mu > 0 \tag{2.7}$$

We rewrite the first two equations of the second vector equation of system (2.3), (2.4) in the form

$$\dot{\hat{\mathbf{p}}} = -G_1\dot{\mathbf{q}} - \partial H/\partial \hat{\mathbf{q}}, \quad \hat{\mathbf{p}} = (p_1, p_2)^T \tag{2.8}$$

where $G_1 = (g_{rj})$ is a $2 \times n$ matrix and $r = 1, 2$. Multiplying respectively the left-hand and right-hand sides of Eq. (2.8) initially by the matrix $(G^2(\mathbf{q}))^{-1}$ and then by the matrix $G_0 = G^2(\mathbf{0})$, we obtain

$$G_0(G^2)^{-1}\dot{\hat{\mathbf{p}}} = -G_0\dot{\mathbf{x}} - G_2\dot{\mathbf{y}} - \partial H/\partial \mathbf{x} + O((\partial H/\partial \mathbf{x})\|\mathbf{q}\|) \tag{2.9}$$

where

$$G_0\dot{\mathbf{x}} + G_2\dot{\mathbf{y}} = G_0(G^2)^{-1}G_1\dot{\mathbf{q}}, \quad \mathbf{x} = \hat{\mathbf{q}} = (q_1, q_2)^T, \quad \mathbf{y} = (q_3, \ldots, q_n)^T$$

In system (2.9) we now make the change of variables

$$G_0(G^2)^{-1}\hat{\mathbf{p}} + G_0\mathbf{x} + G_2\mathbf{y} = G_0\mathbf{z} \tag{2.10}$$

Noting that

$$\mathbf{x} = \mathbf{z} - (G^2)^{-1}\hat{\mathbf{p}} - G_0^{-1}G_2\mathbf{y}$$

and taking relation (2.5) into account, we have, instead of (2.9),

$$G_0\dot{\mathbf{z}} = (G_0(G^2)^{-1})\dot{\hat{\mathbf{p}}} + \dot{G}_2\mathbf{y} + \partial L_0/\partial\mathbf{x}\Big|_{\mathbf{x}\to\mathbf{z}-(G^2)^{-1}\hat{\mathbf{p}}-G_0^{-1}G_2\mathbf{y}} + \mathbf{O}(\|\mathbf{q}\|^m) \tag{2.11}$$

Here $\mathbf{O}(\|\mathbf{q}\|^m)$ denotes a vector, each component of which in the neighbourhood of the point $\mathbf{z} = \mathbf{0}$ is of the order of smallness $\|\mathbf{q}\|^m$.

Since

$$(G_0(G^2)^{-1})^{\boldsymbol{\cdot}} = O(\|\mathbf{p}\|), \quad \dot{G}_2 = O(\|\mathbf{p}\|)$$

where $O(\|\mathbf{p}\|)$ denotes a matrix of the corresponding dimensionality with elements of the order of smallness $\|\mathbf{p}\|$, taking inequalities (2.5) and (2.7) into account, we obtain, instead of (2.11).

$$\begin{gathered} G_0\dot{\mathbf{z}} = \partial L_0^{(m)}(\mathbf{y},\mathbf{z})/\partial\mathbf{z}\Big|_{\mathbf{y}=\mathbf{0}} + \mathbf{O}(\|\mathbf{z}\|^{(m-1+\beta)}) \\ \forall(q,p)\in\Omega, \quad 0<\beta=\text{const} \end{gathered} \tag{2.12}$$

Here, the term $\mathbf{O}(\|z\|^{(m-1+\beta)})$, similar to that described above, is a vector, each component of which in the neighbourhood of the point $z = 0$ is of the order of $\|z\|^{(m-1+\beta)}$. Furthermore,

$$\left(\partial L_0^{(m)}(\mathbf{y},\mathbf{z})/\partial\mathbf{z}\Big|_{\mathbf{y}=\mathbf{0}}\right)\Big|_{\mathbf{z}=\mathbf{0}} = \mathbf{0}$$

$$\partial L_0^{(m)}(\mathbf{y},\mathbf{z})/\partial\mathbf{z}\Big|_{\mathbf{y}=\mathbf{0}} \neq \mathbf{0}, \quad \forall\mathbf{z}\in R^2\setminus\{0\}$$

Since $\det G_0 \neq 0$ then, in accordance with the available results [7] (also see [8] in this connection), the vector equation

$$\varkappa G^2(\mathbf{0})\hat{\mathbf{q}} = \partial L_0^{(m)}/\partial\hat{\mathbf{q}}, \quad \varkappa = \text{const}$$

has a non-trivial, real solution $\hat{\mathbf{q}} = \mathbf{c} \in R^2(\|\mathbf{c}\| \neq 0)$. Hence, a non-singular transformation $\mathbf{v} = B\mathbf{z}$ exists, where B is a constant matrix, which reduces Eqs (2.12) to the form

$$\dot{\mathbf{v}} = \mathbf{V}^{(m-1)}(\mathbf{v}) + \mathbf{O}(\|\mathbf{v}\|^{m-1+\beta}) \tag{2.13}$$

where $\mathbf{V}^{(m-1)}(\mathrm{v}_1, 0) = (a\mathrm{v}_1^{m-1}, 0)^T, a \in \{1, -1\}$.

Next, following Zubov [9], we consider the auxiliary equation

$$\dot{z} = -az^{m-1} \tag{2.14}$$

the solution of which has the form

$$z = z_0[1 + z_0^{(m-2)}(m-2)at]^{-1/(m-2)} \tag{2.15}$$

We shall assume that $z_0 > 0$.

Without loss of generality, we put $a = 1$ in (2.13) since, otherwise, a system with the Lagrangian $L(\mathbf{q}, -\dot{\mathbf{q}})$ could have been considered as the initial system. Making the change of dependent variables in (2.13)

$$\mathbf{v} = z(-\mathbf{e}+\mathbf{w}), \quad \mathbf{e} = (1,0)^T \tag{2.16}$$

instead of (2.13), we have

$$z\dot{\mathbf{w}} = -\dot{z}\mathbf{w} + z^{m-1}F\mathbf{w} + z^{(m-1+\beta)}\mathbf{f}(w_1 - 1, w_2) \tag{2.17}$$

where

$$F = \begin{Vmatrix} f_{11} & f_{12} \\ 0 & f_{22} \end{Vmatrix}$$

is a matrix with constant elements $f_{rs}(r, s = 1, 2)$ and $f_{11} = -m + 1$.

On changing, using (2.15), to the new independent variable, we obtain from (2.17)

$$-z d\mathbf{w}/dz = \mathbf{w} + F\mathbf{w} + z^{\beta}\mathbf{f}(w_1 - 1, w_2) \tag{2.18}$$

The function $\mathbf{f}(w_1 - 1, w_2)$ at the point $\mathbf{w} = 0$ does not vanish in the general situation and, moreover, the number β can also be less than unity. Next, putting

$$\varphi = -\ln z \tag{2.19}$$

and making the substitution $z_1 = z^{\beta}$, we obtain

$$d\mathbf{w}/d\varphi = \mathbf{w} + F\mathbf{w} + \mathbf{b}z_1 + \mathbf{o}(\|(z_1, \mathbf{w})^T\|), \quad dz_1/d\varphi = -\beta z_1 \tag{2.20}$$

where $\mathbf{b}$ is a constant vector and $\mathbf{o}(\|(z_1, \mathbf{w})^T\|)$ denotes a vector with components of a higher order of smallness in the neighbourhood of the origin than $\|(z_1, \mathbf{w})^T\|$.

The corresponding secular equation of the linear approximation of system (2.20)

$$\begin{vmatrix} -m+2-\lambda & f_{12} & b_1 \\ 0 & 1+f_{22}-\lambda & b_2 \\ 0 & 0 & -\beta-\lambda \end{vmatrix} = 0 \tag{2.21}$$

has at least two real negative roots from which, by relations (2.15) and (2.19) and taking account of equality (2.16), we conclude that a solution of system (2.13) exists which tends asymptotically to the point $\mathbf{v} = \mathbf{0}$ when $t \to \infty$. Noting that system (2.13) is the result of a transformation of system (2.3), (2.4) and taking account of the result obtained earlier ([10], a consequence of Lemma 1), we conclude that asymptotic motions to the equilibrium position under investigation $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ exist both when $t \to \infty$ and when $t \to -\infty$. The instability of the equilibrium position being investigated is thereby proved.

Corollary. The equilibrium position $\mathbf{q} = \mathbf{p} = \mathbf{0}$ of a Hamiltonian system with a Hamiltonian of the form

$$H(\mathbf{q},\mathbf{p}) = \sum_{k=2}^{n} \lambda_k(q_k^2 + p_k^2) + H^{(m)}(\mathbf{q},\mathbf{p}) + R(\mathbf{q},\mathbf{p}) \tag{2.22}$$

where $H^{(m)}(\mathbf{q}, \mathbf{p})$ is a homogeneous form of order $m > 2$ and $R(\mathbf{q}, \mathbf{p}) = o(\|\mathbf{q} \oplus \mathbf{p}\|^m)$, $H(\mathbf{q}, \mathbf{p}) \in C^2(D^{2n}_{\mathbf{qp}})$ is unstable if the numbers λ_k are non-zero and of the same sign and the following conditions are also satisfied:

1) $\widehat{(\partial H^{(m)}/\partial \mathbf{z})} \neq \mathbf{0}, \quad \forall \hat{\mathbf{z}} \in R^2\backslash\{0\}$

where $\widehat{(\ldots)}$ and $\hat{\mathbf{z}}$ respectively denote a restriction of the quantity in brackets and the vector $\mathbf{q} \oplus \mathbf{p}$ to the set of zeros of the quadratic form $H^{(2)}(\mathbf{q}, \mathbf{p})$;

2) the functions $\widehat{H^{(m)}}(\mathbf{q}, \mathbf{p})$ do not have an extremum at the point $q_1 = p_1 = 0$; $(\mathbf{q}, \mathbf{p})$

3) $\lim\limits_{\|\mathbf{q}\oplus\mathbf{p}\|\to 0} \|\partial R/\partial(\mathbf{q}\oplus\mathbf{p})\| (\|\mathbf{q}\oplus\mathbf{p}\|)^{-m+l-\alpha} = 0, \quad \text{const} = \alpha > 0.$

It is assumed under the conditions of Theorem 1 that all the coefficients of the quadratic form $L_0^{(2)}(\mathbf{q})$, apart from $\lambda_1 = \lambda_2 = 0$, are negative. It is natural to pose the question: if there are more than two zero numbers among the λ_i, is it possible within the framework of this approach to arrive at any constructive assertion? The answer to this question turns out to be yes.

In accordance with what has been said above, we modify the structure of the function $L_0^{(2)}(\mathbf{q})$ by putting

$$L_0^{(2)}(\mathbf{q}) = \frac{1}{2}\sum_{k=l+1}^{n} \lambda_k q_k^2, \quad l \geqslant 3, \quad \text{const} = \lambda_k < 0 \tag{2.23}$$

Theorem 2. Suppose equalities (2.1) and (2.23) hold and, furthermore,

1) $\partial L_0^{(m)}/\partial q_r \Big|_{\mathbf{q}=(q_1,q_2,0,\dots,0)^T} \neq 0, \quad \forall r = 1,2, \quad (q_1,q_2) \in R^2 \setminus \{0\};$

2) $\partial L_0^{(m)}/\partial q_r \Big|_{\mathbf{q}=(q_1,q_2,0,\dots,0)^T} = 0, \quad \forall r = 3,4,\dots,l;$

3) the function $L_0^{(m)}(q_1, q_2, 0, \dots, 0)$ does not have an extremum at the point $q_1, = q_2, = 0$;

4) $\det G^1(\mathbf{0}) \neq 0$, where $G^1(\mathbf{q})$ is an $l \times l$ matrix which is obtained from the matrix $G(\mathbf{q})$ by cancelling the last $n - l$ rows and the last $n - l$ columns;

5) $\lim\limits_{\|\mathbf{q}\|\to 0} \|\partial R/\partial \mathbf{q}\| (\|\mathbf{q}\|)^{-m+l-\alpha} = 0, \quad \text{const} = \alpha > 0$

Then, the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of system (1.1), (1.2) is unstable.

Proof. We rewrite the first l equations of the second vector equation of system (2.3), (2.4) in the form

$$\dot{\hat{\mathbf{p}}} = -G_1\dot{\mathbf{q}} - \partial H/\partial\hat{\mathbf{q}} \tag{2.24}$$

where $\hat{\mathbf{p}} = (p_1, \dots, p_l)^T$, $G_1 = (g_{rj})$ is an $l \times n$ matrix; $r = 1, \dots, l$; $j = 1, \dots, n$.

Next, following the argument used in the proof of Theorem 1, we arrive at a system of equations of the form of (2.12), in which $G_0 = G^l(\mathbf{0})$, $\mathbf{y} = (q_{l+1}, \dots, q_n)^T$. On the basis of Conditions 1–3 of Theorem 2, we conclude that a non-singular, linear transformation exists which reduces equations of the type (2.12) to the system of l equations

$$\dot{\mathbf{v}} = \mathbf{V}^{(m-1)}(\mathbf{v}) + \mathbf{O}(\|\mathbf{v}\|^{m-1+\beta}) \tag{2.25}$$

where $\mathbf{V}^{(m-1)}(\upsilon_1, 0, \dots, 0) = (a\upsilon_1^{m-1}, 0, \dots, 0)^T$, $a \in \{1, -1\}$.

Then, applying the method used in the proof of Theorem 1, we obtain a system of the type (2.20), the corresponding secular equation of the linear approximation of which has the form

$$\begin{vmatrix} -m+2-\lambda & f_{12} & \dots & f_{1l} & b_1 \\ 0 & 1+f_{22}-\lambda & \dots & f_{2l} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & f_{l2} & \dots & 1+f_{ll}-\lambda & b_l \\ 0 & 0 & \dots & 0 & -\beta-\lambda \end{vmatrix} = 0$$

As before, this secular equation has at least two real negative roots which enables us to conclude that Theorem 2 holds.

Corollary 1. If all the numbers λ_k in expression (2.23) vanish, then Conditions 2 and 4 of Theorem 2 become the following conditions

2*) $\partial L_0^{(m)} / \partial q_r \Big|_{\mathbf{q}=(q_1,q_2,0,\dots,0)^T} = 0, \quad \forall r = 3, 4, \dots, n;$

4*) $\det G(\mathbf{0}) \neq 0$.

Corollary 2. Suppose the function $L_0^{(m)}(\mathbf{q})$ is a homogeneous form of odd degree $m \geqslant 3$. Then, Conditions 1–3 of Theorem 2 can be replaced by the single condition

$$\widehat{\partial L_0^{(m)}/\partial \hat{\mathbf{q}}} \neq \mathbf{0}, \quad \forall \hat{\mathbf{q}} \in R^l \setminus \{0\}$$

where the symbol "$\widehat{\dots}$" denotes a restriction of the quantities $\mathbf{q}$ and $\partial L_0^{(m)}/\partial \hat{\mathbf{q}}$ to the set of zeros of the function $L_0^{(2)}(\mathbf{q})$ defined by equality (2.23).

Proof. The conditions of Corollary 2 always ensure the existence of a non-trivial real solution $\hat{\mathbf{q}} = \mathbf{c} \in R^l(\|\mathbf{c}\| \neq 0)$ of the vector equation

$$\varkappa G^l(\mathbf{0})\hat{\mathbf{q}} = \widehat{\partial L_0^{(m)}/\partial \hat{\mathbf{q}}}, \quad \varkappa = \text{const}$$

which then enables as to use the scheme for the proof of Theorem 1.

Corollary 3. The equilibrium position $\mathbf{q} = \mathbf{p} = \mathbf{0}$ of a Hamiltonian system with a Hamiltonian of the form

$$H(\mathbf{q},\mathbf{p}) = \sum_{k=l+1}^{n} \lambda_k (q_k^2 + p_k^2) + H^{(m)}(\mathbf{q},\mathbf{p}) + R(\mathbf{q},\mathbf{p})$$

$$R(\mathbf{q},\mathbf{p}) = o(\| \mathbf{q} \oplus \mathbf{p} \|^m); \quad H(\mathbf{q},\mathbf{p}) \in C^2(D_{\mathbf{qp}}^{2n})$$

where $H^{(m)}(\mathbf{q}, \mathbf{p})$ is a homogeneous form of odd order $m \geqslant 3$, is unstable if the numbers $\lambda_k \neq 0$ have the same sign and the following conditions are satisfied

1) $\widehat{(\partial H^{(m)}/\partial(\widehat{\mathbf{q}\oplus\mathbf{p}}))} \neq \mathbf{0}, \ \forall (\widehat{\mathbf{q}\oplus\mathbf{p}}) \in R^{2l} \setminus \{0\}$

where the symbol "$\widehat{\dots}$" denotes a restriction of the magnitudes of $\mathbf{q} \oplus \mathbf{p}$ and $\partial H^{(m)}/\partial(\widehat{\mathbf{q}\oplus\mathbf{p}})$ to the set of zeros of the quadratic form $H^{(2)}(\mathbf{q}, \mathbf{p})$;

2) $\lim\limits_{\|\mathbf{q}\oplus\mathbf{p}\| \to 0} \| \partial R / \partial(\mathbf{q} \oplus \mathbf{p}) \| (\| \mathbf{q} \oplus \mathbf{p} \|)^{-m+1-\alpha} = 0, \quad \text{const} = \alpha > 0.$

3. THE INSTABILITY OF SYSTEMS WITH TWO DEGREES OF FREEDOM

Theorems 1 and 2 presuppose that the Lagrangian has a very special structure which holds in far from in all cases of gyroscopic systems. In particular, in the proof of Theorems 1 and 2 (apart from the case when it is required that the number m is odd), it is essential that a certain subsystem with two degrees of freedom may separate out in the initial system which would then enable the stability investigation to be reduced to a certain standard situation. In this connection, it is of interest to consider in greater detail systems with two degrees of freedom expecting the stronger results to be obtained in this case.

To investigate the stability, we use Hamiltons' action function [10–12] (the prime denotes a time derivative)

$$S = \int_0^t L(\mathbf{q}, \mathbf{q}')d\tau \tag{3.1}$$

Assuming that the solutions of system (1.1), (1.2) with the origin at $D_\mathbf{q} \times R^n$ are extendable along the whole axis $t \in R$, we can represent the action function in the form

$$S = S^*(\tau, \mathbf{q}(\tau), \mathbf{q}'(\tau))\big|_0^t \tag{3.2}$$

The assumption regarding the extendability leads to no loss of generality in the treatment if account is taken of the fact that the following discussion is concerned with the instability of an equilibrium.

Theorem 3. Suppose $n = 2$ and a number $\varepsilon > 0$ ($D_\mathbf{q} \supset \bar{s}_\varepsilon$) exists for which the following conditions are satisfied

1) $\omega = \{\mathbf{q} \in s_\varepsilon = \{\mathbf{q} \in D_\mathbf{q}, \|\mathbf{q}\| < \varepsilon\} : L_0(\mathbf{q}) > 0\} \neq \emptyset, \quad 0 \in \partial\omega$;
2) the set ω does not contain the cycle encircling the point $\mathbf{q} = \mathbf{0}$;
3) $\det G(\mathbf{0}) \neq 0$;
4) $\partial L_0 / \partial \mathbf{q} \neq \mathbf{0} \quad \forall \mathbf{q} \in s_\varepsilon \setminus 0$.

The equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of system (1.1), (1.2) is then unstable.

Proof. Under the conditions of Theorem 3, when account is taken of Darboux's theorem (see [7, 13]), generalized coordinates can always be chosen such that

$$L_1 = (-q_2\dot{q}_1 + q_1\dot{q}_2)$$

We represent Eqs (1.1) in the Hamiltonian form

$$\begin{aligned} &\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} \\ &H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\mathbf{p}^T A^{-1}\mathbf{p} - \mathbf{p}^T A^{-1}\mathbf{f} - L_0 + \frac{1}{2}\mathbf{f}^T A^{-1}\mathbf{f} = h = \text{const} \end{aligned} \tag{3.3}$$

Taking (3.3) into account, we rewrite the action function S, which is defined by equality (3.2), in the form

$$S_1(t, \mathbf{q}, \mathbf{p}) = S^*(\tau, \mathbf{q}(\tau), \frac{\partial H}{\partial \mathbf{p}}(\tau))\big|_0^t = S_1^*(\tau, \mathbf{q}(\tau), \mathbf{p}(\tau))\big|_0^t$$

In accordance with the initial assumptions and the conditions of Theorem 3 $S_1^* \in C^1(R \times s_\varepsilon^*)$.
We now consider the set

$$\Omega^* = \{(\mathbf{q}, \mathbf{p}) \in s_\varepsilon^* = \{(\mathbf{q}, \mathbf{p}) \in D_\mathbf{q} \times R^n, \quad \|\mathbf{q} \oplus \mathbf{p}\| < \varepsilon\} : H(\mathbf{q}, \mathbf{p}) = h = 0\}$$

The set Ω^* is non-empty on the basis of Condition 1 of Theorem 3.

We assume that the equilibrium position $\mathbf{q} = \mathbf{p} = \mathbf{0}$ of system (3.3) is stable and we denote the set of positive limit points of its trajectories by Λ^+. By the assumption of stability $\Lambda^+ \cap \Omega^* \setminus \{0, 0\} = \Lambda_0^+ \neq \emptyset$. The set Λ_0^+, as it consists of the positive limit points of the trajectories belonging to Ω^*, is compact and, therefore, contains the minimum set $\Gamma(\mathbf{q}, \mathbf{p}) \subset \Lambda_0^+$ [14, p. 401]. This last set is distinct from the equilibrium state according to Condition 4 of Theorem 3. The set Γ is also compact since it is a closed subset of the set Λ_0^+. Hence, in accordance with Birkhoff's theorem [14, p. 402], we conclude that any trajectory $\gamma \subset \Gamma$ is recurrent and thereby stable in Poisson's sense [14, p. 363], that is, a sequence $\{t_k\}$ ($k = 1, 2, \ldots$) exists such that

$$\lim_{k\to\infty} t_k = \infty, \quad \lim_{k\to\infty} \| \mathbf{q}(t_k, \mathbf{q}_0, \mathbf{p}_0) \oplus \mathbf{p}(t_k, \mathbf{q}_0, \mathbf{p}_0) \| = \| \mathbf{q}_0 \oplus \mathbf{p}_0 \|, \quad \forall (\mathbf{q}_0, \mathbf{p}_0) \in \gamma$$

Together with (3.3), we now consider the auxiliary system

$$\dot{\mathbf{q}} = \frac{1}{\mathbf{q}^2}\frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{1}{\mathbf{q}^2}\frac{\partial H}{\partial \mathbf{q}}; \quad \mathbf{q}^2 = q_1^2 + q_2^2 \tag{3.4}$$

Since $\Gamma \subset \Omega^*\backslash\{0, 0\}$, the expression $\gamma \subset \Gamma$ does not vanish on any trajectory $\mathbf{q}^2$. Hence, if one confines oneself solely to a consideration of the set of trajectories Γ, neglecting the velocities of motion of corresponding representative points along these trajectories, then, when Condition 2 of the theorem is taken into account, we conclude that auxiliary system (3.4) is equivalent to system (3.3).

We now consider the derivative of the function $S_1(t, \mathbf{q}, \mathbf{p})$ of the vector field with respect to t, which is defined by Eqs (3.4). As a result, we obtain

$$\frac{dS_1}{dt} = \frac{\partial S_1}{\partial t} + \frac{1}{\mathbf{q}^2}\left[\frac{\partial S_1}{\partial \mathbf{q}}\frac{\partial H}{\partial \mathbf{p}} - \frac{\partial S_1}{\partial \mathbf{p}}\frac{\partial H}{\partial \mathbf{q}}\right] \tag{3.5}$$

Since, by Lemma 2 from [10],

$$\partial S_1 / \partial t = 0, \quad \forall (\mathbf{q}, \mathbf{p}) \in \Omega^* \tag{3.6}$$

equality (3.5) can be rewritten in the form

$$\frac{dS_1}{dt} = \frac{1}{\mathbf{q}^2} L\left(\mathbf{q}, \frac{\partial H}{\partial \mathbf{p}}\right) = \frac{L(\mathbf{q}, \dot{\mathbf{q}})}{\mathbf{q}^2} = \frac{(L_2 + L_0)}{\mathbf{q}^2} + \frac{(-q_2\dot{q}_1 + q_1\dot{q}_2)}{\mathbf{q}^2} \tag{3.7}$$

We integrate equality (3.7) along a segment of the positive half-trajectory which is stable in Poisson's sense $\gamma_{0,k}^+ \subset \Gamma(\gamma_{0,k}^+ = \{\mathbf{q}(t), \mathbf{p}(t) : t \in [0, t_k] \subset R^+, t_k \in \{t_k\}\}]$ of system (3.4). As a result, we obtain

$$\int_{\gamma_{0,k}^+} dS_1 = \int_0^{t_k} \frac{2}{\mathbf{q}^2} L_2 dt + \varphi|_0^{t_k}, \quad \varphi = \arccos\frac{q_1}{\|\mathbf{q}\|} \tag{3.8}$$

According to Condition 2 of the Theorem 3, the second term on the right-hand side of equality (3.8) is bounded. As far as the first term is concerned, noting that the expression for L_2 is non-negative and that $L_2 \not\equiv 0$ in any trajectory passing through $\Omega^*\backslash\{0, 0\}$, by the mode of reasoning given in [10], we arrive at the conclusion that, when $t_k \to \infty$, it is unbounded from above in $\gamma_{0,k}^+$. At the same time, since the function S_1 does not have any singularities in s_ε^*, the result of integration of the left-hand side of equality (3.8) can be represented in the form

$$\int_{\gamma_{0,k}^+} dS_1 = S_1^*(t_k, \mathbf{q}(t_k), \mathbf{p}(t_k)) - S_1^*(0, \mathbf{q}(0), \mathbf{p}(0)) \tag{3.9}$$

Since values of the function $(\mathbf{q}(t_k), \mathbf{p}(t_k))$ which are bounded from above correspond, when account is taken of (3.6), to a point S_1^* which returns to the initial point $(\mathbf{q}(0), \mathbf{p}(0))$ when $t_k \to \infty$, we conclude on the basis of relations (3.9) that equality (3.8) is contradictory. Consequently, the assumption regarding the stability of the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of system (1.1), (1.2) is untrue. Theorem 3 is proved.

Corollary. Under the conditions of Theorem 3, solutions of system (1.1), (1.2) exist which asymptotically tend to the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ when $t \to \infty$ and $t \to -\infty$.

Proof. It follows from the scheme for the proof of Theorem 3 that the set $\Omega^*\backslash\{0, 0\}$ does not contain whole trajectories. Hence, using the approach which has been described previously in [10, p. 92], we conclude that the corollary holds.

Theorem 3 generalizes the results concerning systems with two degrees of freedom obtained previously in [7, 15].

Remark. Condition 2 in Theorem 3 is essential as can be seen from the example of the system

$$\ddot{q}_1 - 2\lambda\dot{q}_2 = \frac{\partial L_0}{\partial q_1}, \quad \ddot{q}_2 + 2\lambda\dot{q}_1 = \frac{\partial L_0}{\partial q_2}; \quad \text{const} = \lambda \neq 0$$
$$L_0(\mathbf{q}) = L_0(\mathbf{q}^2) \in C^2 \tag{3.10}$$

where the point $q_1 = q_2 = 0$, as above, corresponds to an equilibrium position.

As a result of the change of variables

$$x = \cos\lambda t q_1 - \sin\lambda t q_2, \quad y = \sin\lambda t q_1 + \cos\lambda t q_2 \tag{3.11}$$

which satisfies the equality: $\mathbf{q}^2 = x^2 + y^2$, instead of (3.10), we obtain

$$\ddot{x} + \lambda^2 x = \frac{\partial L_0}{\partial x}, \quad \ddot{y} + \lambda^2 y = \frac{\partial L_0}{\partial y}$$
$$L_0(x^2 + y^2) = L_0(q_1^2 + q_2^2) \tag{3.12}$$

Hence, as a consequence of transformation (3.11), system (3.10) becomes the natural system (3.12), the energy integral of which takes the form

$$\frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}\lambda^2(x^2 + y^2) - L_0(x^2 + y^2) = h = \text{const} \tag{3.13}$$

Since $L_0 \in C^2, x = y = 0$ in the neighbourhood of the point $L_0(x^2 + y^2) = O(x^2 + y^2)$ as the equilibrium position of the system. Hence, owing to the choice of the constant λ, it is always possible to ensure that the potential energy

$$\Pi = \frac{1}{2}\lambda^2(x^2 + y^2) - L_0(x^2 + y^2)$$

of the natural system which has been obtained has a strict local minimum at the point $x = y = 0$, regardless of the value of $L_0(x^2 + y^2)$, and that the equilibrium position $x = y = 0$ is thereby stable. It is obvious that the possibility of Condition 2 of Theorem 3 being satisfied is precluded by the structure of the function $L_0(x^2 + y^2)$.

Example. We will now consider the problem of the stability of the steady motions of an artificial satellite which is a dynamically symmetric rigid body, the central ellipsoid of inertia of which is an ellipsoid of revolution [2, p. 92]. The Lagrange function has the form

$$L = \frac{1}{2}A[\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta + 2\omega_0(\dot{\theta}\sin\psi + \dot{\psi}\sin\theta\cos\theta\cos\psi) - \omega_0^2\sin^2\theta\cos^2\psi] +$$
$$+\frac{1}{2}C(\dot{\varphi} + \dot{\psi}\cos\theta - \omega_0\sin\theta\cos\psi)^2 - \frac{3}{2}\omega_0^2(C - A)\cos^2\theta$$

Here A and C are the principal central moments of inertia of the satellite; θ, ψ, φ are the Euler angles by means of which the position of the satellite in the orbital system of coordinates is defined, and ω_0 is the angular velocity of the centre of mass of the satellite in its motion in the orbit.

Since the angle of natural rotation φ is a cyclic coordinate, we have

$$\frac{\partial L}{\partial\dot{\varphi}} = C(\dot{\varphi} + \dot{\psi}\cos\theta - \omega_0\sin\theta\cos\psi) = C\omega_3^0 = \text{const} \tag{3.14}$$

Integral (3.14) reflects the constancy of the projection of the instantaneous angular velocity of the satellite onto its axis of dynamic symmetry.

Eliminating the cyclic coordinate, we obtain Routh's function

$$R = \frac{1}{2}A[\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta + 2\omega_0(\dot{\theta}\sin\psi + \dot{\psi}\sin\theta\cos\theta\cos\psi)] + C\omega_3^0\dot{\psi}\cos\theta - W$$
$$W = \frac{3}{2}\omega_0^2(C - A)\cos^2\theta + \frac{1}{2}A\omega_0^2\sin^2\theta\cos^2\psi + C\omega_3^0\omega_0\sin\theta\cos\psi + \frac{1}{2}C\omega_3^{02}$$

(W is the reduced potential energy).

The relative steady motions of a satellite, the centre of mass of which moves uniformly in a circular orbit, are determined from the equations

$$\frac{\partial W}{\partial \theta}=\frac{\partial W}{\partial \psi}=0 \tag{3.15}$$

We will confine ourselves to considering one of a series of solutions of system (3.15) [2, p. 94]

$$\theta_0=\frac{\pi}{2},\quad \cos\psi_0=-\frac{C\omega_3^0}{A\omega_0} \tag{3.16}$$

On putting, in the perturbed motion,

$$\theta=\theta_0+x,\ \psi=\psi_0+y$$

in a small neighbourhood of solution (3.16), we have

$$\begin{aligned}
&W(\theta,\psi)-W\left(\frac{\pi}{2},\psi_0\right)=\frac{3}{2}\omega_0^2(C-A)\sin^2 x+\frac{1}{2}A\omega_0^2(1-\cos^2\psi_0)\sin^2 y+\\
&+\frac{1}{2}A\omega_0^2\cos\psi_0\sin\psi_0\sin y(\sin^2 x+\sin^2 y)+\frac{1}{8}\frac{C^2}{A}\omega_3^{0 2}(\sin^4 x+\sin^4 y)+\\
&+\frac{1}{2}\left[-A\omega_0^2(1-\cos^2\psi_0)+\frac{1}{2}\frac{C^2}{A}\omega_3^{0 2}\right]\sin^2 x\sin^2 y+o[(\sin^2 x+\sin^2 y)^2]
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
&R_1(\dot\theta,\ \dot\psi,\ \theta,\ \psi)-R_1\left(0,\ 0,\ \frac{\pi}{2},\ \psi_0\right)=A\omega_0\cos\psi_0 y\dot x-\\
&-(A\omega_0\cos\psi_0+C\omega_3^0)x\dot y+O(x^2+y^2)(|\dot x|+|\dot y|)
\end{aligned}$$

In addition to the result which has been obtained previously [2, p. 97], we consider the case when the Poincaré stability coefficients can vanish. From Theorem 3, taking account of equality (3.17) and, also, the fact that $g_{12}(0,0)=-g_{21}(0,0)=-C\omega_3^0$ in the problem being considered, we conclude that steady motion (3.16) is unstable in the following cases

1) $A>C$, $|\cos\psi_0|=1$;
2) $A=C$, $0<|\cos\psi_0|<1$.

On the other hand, by Routh's theorem, steady motion (3.16) is stable if any of the following conditions hold

1) $A<C$, $|\cos\psi_0|\leqslant 1$;
2) $A=C$, $|\cos\psi_0|=1$.

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